Rang matrice

Minor reda k matrice A je determinanta reda k sastavljena od elemenata koji stoje na presjecima proizvoljnih k vrsta i k kolona matrice A.

Npr.
$$A=\begin{bmatrix}3&4&5&6&7\\1&2&3&4&5\\3&4&7&5&2\\2&3&1&7&5\end{bmatrix}$$

minor reda 3
$$\begin{vmatrix}4&5&6\\2&3&4\\4&7&5\end{vmatrix}$$

minor reda 4
$$\begin{vmatrix}4&5&6&7\\2&3&4&5\\4&7&5&2\\3&1&7&5\end{vmatrix}$$

Rang matrice A je broj (**označavamo ga sa rang(A)**) koji je jednak redu maksimalnog minora, različitog od nule, determinante $\det A$.

Za dvije matrice A i B kažemo da su ekvivalentne ako imaju isti rang. Rang matrice tražimo elementarnim transformacijama:

1. razmjena mjesta dvije vrste (red) ili dvije kolone
2. dodavanjem elemenata jednog reda (ili jedne kolone) odgovarajućim elementima drugog reda (ili druge kolone) pomnožena nekim brojem.
3. množenje elemenata jednog reda (ili jedne kolone) nekim brojem različitim od nule

Ekvivalentne matrice označavamo sa $A\sim B$.

1. Odrediti rang matrice:

a)
$$M=\begin{bmatrix}2&-1&3&-2&4\\4&-2&4&1&7\\2&-1&1&8&2\end{bmatrix}$$

Rj.
$$\begin{bmatrix}2&-1&3&-2&4\\4&-2&4&1&7\\2&-1&1&8&2\end{bmatrix}\underset{III_R-I_R}{\overset{II_R-I_R\cdot 2}{\sim}}\begin{bmatrix}2&-1&3&-2&4\\0&0&-2&5&-1\\0&0&-2&10&-2\end{bmatrix}\overset{III_R-II_R}{\sim}\begin{bmatrix}2&-1&3&-2&4\\0&0&-2&5&-1\\0&0&0&5&-1\end{bmatrix},\quad \text{rang}(M)=3$$

b)
$$A=\begin{bmatrix}-2&1&0&2\\0&-1&1&3\\-1&1&0&-2\\-4&2&1&1\end{bmatrix}$$

Rj.
$$\begin{bmatrix}-2&1&0&2\\0&-1&1&3\\-1&1&0&-2\\-4&2&1&1\end{bmatrix}\overset{I_k\leftrightarrow II_k}{\sim}\begin{bmatrix}1&-2&0&2\\-1&0&1&3\\1&-1&0&-2\\2&-4&1&1\end{bmatrix}\underset{\substack{III_R-I_R\\IV_R-I_R\cdot 2}}{\overset{II_R+I_R}{\sim}}\begin{bmatrix}1&-2&0&2\\0&-2&1&5\\0&1&0&-4\\0&0&1&-3\end{bmatrix}\overset{II_k\leftrightarrow III_k}{\sim}$$

$$\sim\begin{bmatrix}1&0&-2&2\\0&1&-2&5\\0&0&1&-4\\0&1&0&-3\end{bmatrix}\overset{IV_R-I_R}{\sim}\begin{bmatrix}1&0&-2&2\\0&1&-2&5\\0&0&1&-4\\0&0&2&-8\end{bmatrix}\overset{IV_R-III_R\cdot 2}{\sim}\begin{bmatrix}1&0&-2&2\\0&1&-2&5\\0&0&1&-4\\0&0&0&0\end{bmatrix},\quad \text{rang}(A)=2$$

2. Odrediti rang matrice $A=\begin{bmatrix}1&1&1&1\\2&3&-1&1\\3&4&0&\lambda+2\end{bmatrix}$, $\lambda\in\mathbb{R}$.

Rj.
$$\begin{bmatrix}1&1&1&1\\2&3&-1&1\\3&4&0&\lambda+2\end{bmatrix}\overset{III_k\leftrightarrow I_k}{\sim}\begin{bmatrix}1&1&1&1\\-1&3&2&1\\0&4&3&\lambda+2\end{bmatrix}\overset{II_v+I_v}{\sim}\begin{bmatrix}1&1&1&1\\0&4&3&2\\0&4&3&\lambda+2\end{bmatrix}\overset{III_v-II_v}{\sim}$$

$$\sim\begin{bmatrix}1&1&1&1\\0&4&3&2\\0&0&0&\lambda\end{bmatrix},$$

ako je $\lambda=0$ tada je $\text{rang}(A)=2$

ako je $\lambda\neq 0$ tada je $\text{rang}(A)=3$

3. U ovisnosti o parametru $\lambda\in\mathbb{R}$ odredite rang matrice
$$A=\begin{bmatrix}1&1&1\\1&\lambda&\lambda^2\\1&\lambda^2&\lambda\end{bmatrix}.$$

Rj.
$$\begin{bmatrix}1&1&1\\1&\lambda&\lambda^2\\1&\lambda^2&\lambda\end{bmatrix}\underset{III_R-I_R}{\overset{II_R-I_R}{\sim}}\begin{bmatrix}1&1&1\\0&\lambda-1&\lambda^2-1\\0&\lambda^2-1&\lambda-1\end{bmatrix}\underset{\substack{III_R:(\lambda-1)\\ \lambda\neq 1\\ \text{zapamtimo}}}{\overset{II_R:(\lambda-1)}{\sim}}\begin{bmatrix}1&1&1\\0&1&\lambda+1\\0&\lambda+1&1\end{bmatrix}\overset{III_R+II_R\cdot(-\lambda-1)}{\sim}\begin{bmatrix}0&1&1\\0&1&\lambda+1\\0&0&\underbrace{-(\lambda+1)^2+1}_{-(\lambda^2+2\lambda+1)+1}\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&1&\lambda+1\\0&0&-\lambda(\lambda+2)\end{bmatrix}$$

Matrica se ne može više pojednostaviti. Diskusija:

Za $\lambda=0$ dobićemo $\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}\Rightarrow \text{rang}\,A=2$

Za $\lambda=-2$ imamo $\begin{bmatrix}1&1&1\\0&1&-1\\0&0&0\end{bmatrix}\Rightarrow \text{rang}\,A=2$.

Ostaje nam još slučaj $\lambda=1$. Zašto?

Za $\lambda=1$, $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\Rightarrow \text{rang}\,A=1$. Zašto?

U ostalim slučajevima (tj. kad je $\lambda\neq 0$, $\lambda\neq -2$ i $\lambda\neq 1$) $\text{rang}\,A=3$.

4. Diskutovati rang matrice $M=\begin{bmatrix}1&10&-6&\lambda\\2&-1&\lambda&3\\1&\lambda&-1&2\end{bmatrix}$.

5. Diskutovati o rangu matrice
$$M=\begin{bmatrix}a&b&1\\1&ab&1\\1&b&a\end{bmatrix}$$
u zavisnosti od parametara a i b.

Diskutovati rang matrice u zavisnosti od parametara a i b,

$$A=\begin{bmatrix}2&3&9&6&2\\5&4&12&8&5\\1&2&6&4&1\\4&1&3&2&a\\3&b&6&4&3\\7&5&15&10&7\end{bmatrix}$$

Rj.

$$A=\begin{bmatrix}2&3&9&6&2\\5&4&12&8&5\\1&2&6&4&1\\4&1&3&2&a\\3&b&6&4&3\\7&5&15&10&7\end{bmatrix}\overset{II_k\leftrightarrow IV_k}{\sim}\begin{bmatrix}2&6&9&3&2\\5&8&12&4&5\\1&4&6&2&1\\4&2&3&1&a\\3&4&6&b&3\\7&10&15&5&7\end{bmatrix}\overset{IV_R\leftrightarrow VI_R}{\sim}\begin{bmatrix}2&6&9&3&2\\5&8&12&4&5\\1&4&6&2&1\\7&10&15&5&7\\3&4&6&b&3\\4&2&3&1&a\end{bmatrix}\overset{III_R\leftrightarrow I_R}{\sim}$$

$$\sim\begin{bmatrix}1&4&6&2&1\\5&8&12&4&5\\2&6&9&3&2\\7&10&15&5&7\\3&4&6&b&3\\4&2&3&1&a\end{bmatrix}\begin{matrix}II_R-I_R\cdot 5\\III_R-I_R\cdot 2\\ \sim\\IV_R-I_R\cdot 7\\V_R-I_R\cdot 3\\VI_R-I_R\cdot 4\end{matrix}\begin{bmatrix}1&4&6&2&1\\0&-12&-18&-6&0\\0&-2&-3&-1&0\\0&-18&-27&-9&0\\0&-8&-12&b-6&0\\0&-14&-21&-7&a-4\end{bmatrix}\overset{II_k\leftrightarrow V_k}{\sim}\begin{bmatrix}1&1&6&2&4\\0&0&-18&-6&-12\\0&0&-3&-1&-2\\0&0&-27&-9&-18\\0&0&-12&b-6&-8\\0&a-4&-21&-7&-14\end{bmatrix}\sim$$

$$\overset{II_R\leftrightarrow III_R}{\sim}\begin{bmatrix}1&1&6&2&4\\0&0&-3&-1&-2\\0&0&-18&-6&-12\\0&0&-27&-9&-18\\0&0&-12&b-6&-8\\0&a-4&-21&-7&-14\end{bmatrix}\begin{matrix}III_R-II_R\cdot 6\\ \sim\\IV_R-II_R\cdot 9\\V_R-II_R\cdot 4\\VI_R-II_R\cdot 7\end{matrix}\begin{bmatrix}1&1&6&2&4\\0&0&-3&-1&-2\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&b-2&0\\0&a-4&0&0&0\end{bmatrix}$$

Diskusija

1° $a=4$, $b=2$ $\quad$ $\text{rang}\,A=2$

2° $a=4$, $b\neq 2$ $\quad$ $\text{rang}\,A=3$

3° $a\neq 4$, $b=2$ $\quad$ $\text{rang}\,A=3$

4° $a\neq 4$, $b\neq 2$ $\quad$ $\text{rang}\,A=4$

Diskutovati rang matrice

$$M=\begin{bmatrix}14&4&2\lambda-4&-6\\6&2&-1&-3\\3\lambda+4&2&-2\lambda+1&-3\\24&8&-4&-12\end{bmatrix}$$

za razne vrijednosti parametra λ.

Rj.

$$M=\begin{bmatrix}14&4&2\lambda-4&-6\\6&2&-1&-3\\3\lambda+4&2&-2\lambda+1&-3\\24&8&-4&-12\end{bmatrix}\overset{III_v+I_v}{\sim}\begin{bmatrix}14&4&2\lambda-4&-6\\6&2&-1&-3\\3\lambda+18&6&-3&-9\\24&8&-4&-12\end{bmatrix}\begin{matrix}IV_v:4\\I_v:2\\III_v:3\end{matrix}$$

$$\begin{bmatrix}7&2&\lambda-2&-3\\6&2&-1&-3\\\lambda+6&2&-1&-3\\6&2&-1&-3\end{bmatrix}\overset{IV_v-II_v}{\sim}\begin{bmatrix}7&2&\lambda-2&-3\\6&2&-1&-3\\\lambda+6&2&-1&-3\\0&0&0&0\end{bmatrix}\overset{I_v\leftrightarrow II_v}{\sim}$$

$$\begin{bmatrix}6&2&-1&-3\\7&2&\lambda-2&-3\\\lambda+6&2&-1&-3\\0&0&0&0\end{bmatrix}\overset{I_k\leftrightarrow IV_k}{\sim}\begin{bmatrix}-3&2&-1&6\\-3&2&\lambda-2&7\\-3&2&-1&\lambda+6\\0&0&0&0\end{bmatrix}\begin{matrix}II_v-I_v\\ \sim\\III_v-I_v\end{matrix}$$

$$\sim\begin{bmatrix}-3&2&-1&6\\0&0&\lambda-1&1\\0&0&0&\lambda\\0&0&0&0\end{bmatrix}$$

Za $\lambda=0$ $\quad\text{rang}(M)=2$

Za $\lambda\neq 0$ $\quad\text{rang}(M)=3$

Diskutovati rang matrice $\begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix}$ za razne vrijednosti parametra t.

Rj.

$$M = \begin{bmatrix} 1 & 2 & t & 0 & -1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 4 & -2 \\ 1 & 0 & 0 & -3 & 4 \end{bmatrix} \overset{III_k \leftrightarrow V_k}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 & t \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 0 & 4 & -3 & 0 \end{bmatrix} \overset{I_v \leftrightarrow IV_v}{\sim}$$

$$\begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 1 & 2 & -1 & 0 & t \end{bmatrix} \underset{IV_v - I_v}{\overset{II_v - I_v \cdot 2}{\sim}} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix} \overset{II_v \leftrightarrow III_v}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 2 & -5 & 3 & t \end{bmatrix}$$

$$\overset{IV_v + II_v \cdot 2}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & -9 & 11 & t \end{bmatrix} \overset{IV_v - III_v \cdot \frac{3}{2}}{\sim} \begin{bmatrix} 1 & 0 & 4 & -3 & 0 \\ 0 & -1 & -2 & 4 & 0 \\ 0 & 0 & -6 & 7 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & t \end{bmatrix}$$

$$-9 + 6 \cdot \frac{3}{2} = -9 + 9 = 0$$

$$11 - 7 \cdot \frac{3}{2} = \frac{22}{2} - \frac{21}{2} = \frac{1}{2}$$

Bez obzira na vrijednost parametra t rang matrice M je uvijek 4.